ACR No. L6A22

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS Copy 2

WARTIME REPORT

ORIGINALLY ISSUED
March 1946 as
Advance Confidential Report L6A22

APPROXIMATE FORMULAS FOR THE COMPUTATION OF TURBULENT BOUNDARY-LAYER MOMENTUM THICKNESSES IN COMPRESSIBLE FLOWS

By Neal Tetervin

Langley Memorial Aeronautical Laboratory
Langley Field, Va.

NACA

WASHINGTON

L - 119

NACA ACR No. L6A22

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

ADVANCE CONFIDENTIAL REPORT

APPROXIMATE FORMULAS FOR THE COMPUTATION OF TURBULENT BOUNDARY-LAYER MOMENTUM THICKNESSES IN COMPRESSIBLE FLOWS

By Neal Tetervin

SUMMARY

Approximate formulas for the computation of the momentum thicknesses of turbulent boundary layers on two-dimensional bodies, on bodies of revolution at zero angle of attack, and on the inner surfaces of round channels all in compressible flow are given in the form of integrals that can be conveniently computed. The formulas involve the assumptions that the momentum thickness may be computed by use of a boundary-layer velocity profile which is fixed and that skin-friction formulas for flat plates may be used in the computation of boundary-layer thicknesses in flow with pressure gradients. The effect of density changes on the ratio of the displacement thickness to the momentum thickness of the boundary layer is taken into account. Use is made of the experimental finding that the skin-friction coefficient for turbulent flow is independent of Mach number. The computations indicate that the effect of density changes on the momentum thickness in flows with pressure gradients is small for subsonic flows.

INTRODUCTION

A number of methods are available for the computation of boundary-layer momentum thicknesses for incompressible flow. The increasing importance of flows at Mach numbers approaching and exceeding 1 has emphasized the need of formulas that would make possible the comparatively rapid computation of boundary-layer momentum thicknesses for compressible flows. The purpose of the present work is therefore to provide approximate formulas for the computation of boundary-layer-thickness parameters for compressible flows. The present work furnishes no new information concerning the boundary-layer shape, skin-friction coefficient, position of the transition point, or likelihood of boundary-layer separation.

Approximate formulas for the computation of the momentum thicknesses of turbulent boundary layers on two-dimensional bodies, on bodies of revolution at zero angle of attack, and on the inner surfaces of round channels all in compressible flow are given in the form of integrals that can be conveniently computed. The approximate formulas contain the assumptions that the momentum thickness may be computed by use of a boundary-layer velocity profile which is fixed during the integration and that skin-friction formulas for flat plates may be used in the computation of boundary-layer momentum thicknesses for flow with pressure gradients. The formulas are applicable to all unseparated, turbulent boundary layers and in special cases to laminar boundary layers. The numerical values of the ratio of the displacement thickness to the momentum thickness, a ratio that appears in the momentum equation and that is capable of specifying approximately the velocity distribution through the turbulent boundary layer in incompressible flow, are corrected for density changes in the boundary layer by use of low-speed velocity distributions. Use is made of the experimental finding that the skin-friction coefficient in turbulent flow is independent of Mach number.

The problem of computing the boundary-layer momentum thickness for compressible flow has been treated by Young and Winterbottom (reference 1), who integrated the boundary-layer momentum equation for laminar flow by using the skin-friction relation from the Pohlhausen theory (reference 2, p. 109), fixing the velocity profile, and correcting the density through the boundary layer for the effects of compressibility. For the turbulent boundary layer, the momentum equation was integrated by a step-by-step process in which a fixed velocity profile was used and the effect of density changes through the boundary layer on the ratio of the displacement thickness to the momentum thickness was ignored.

The problem of computing the momentum thickness over a body of revolution for incompressible flow has been treated by Young (reference 3), who computed the momentum thickness of the laminar boundary layer by a step-by-step computation in which an extension of the Pohlhausen method was used. The thickness of the turbulent part of the boundary layer was computed by a step-by-step process in which a fixed velocity profile and the momentum equation for a body of revolution were used.

In order to substantiate the assumption that skin-friction formulas for turbulent flow along flat plates may be used in the computation of momentum thicknesses for flow with pressure gradients, references 3 to 6 are cited. In these references, good agreement between calculated and experimental results was generally obtained although fairly large adverse pressure gradients were present in many of the cases.

The assumption that the momentum thickness may be computed to a close approximation by fixing the velocity profile during integration is substantiated by the work in references 3 and 4 and by that in reference 6, which contains a comparison between the computed and experimental values of the momentum thickness over the entire chord.

That the skin-friction coefficient for turbulent flow is independent of Mach number is established by the work in references 7 and 8. Frössel (reference 7) presents experimental data for turbulent flow in pipes which show that the velocity profiles for subsonic compressible flow and the skin-friction coefficients for subsonic and supersonic compressible flow do not differ noticeably from those for incompressible flow. Theodorsen and Regier (reference 8), by experimenting with rotating disks, showed that the skin-friction coefficient for turbulent boundary layers is independent of Mach number. Keenan and Neuman (reference 9), after performing experiments with pipes, reached conclusions that did not contradict those of references 7 and 8.

SYMBOLS

a constant in equation relating H_c to λ and H_i

b slope of velocity distribution

c_d drag coefficient per unit span

c_o	velocity of sound in free stream
c_p	specific heat at constant pressure, foot-pounds per pound-mass per degree
H	ratio of displacement thickness to momentum thickness (δ^*/θ)
K	constant
k	constant in skin-friction formula
L	length of airfoil, body of revolution, or round channel; measured along chord or axis of revolution
M_o	free-stream Mach number
m	exponent in formula for boundary-layer velocity distribution
m'	particular value of m
n	exponent in skin-friction formula
p	static pressure
R	gas constant
R_L	Reynolds number (U_oL/υ_o)
R_x	Reynolds number based on length of plate (U_ox/υ_o)
R_θ	Reynolds number based on momentum thickness $(U\theta/\upsilon)$
r	radial distance of point from axis of body of revolution or round channel
r_t	radius of body of revolution or round channel
$r_{t_{max}}$	maximum radius of body of revolution or round channel
T	absolute temperature
T_o	absolute temperature of free stream
T_δ	absolute temperature at edge of boundary layer

U velocity parallel to x at outer edge of boundary layer

U_0 free-stream velocity

U_x value of U at station at which value of $\left(\frac{\theta}{L}\right)_x$ is obtained

U_{x_0} value of U at $x = 0$

U_λ value of U chosen to make value of $\left(\frac{U_\lambda}{U_x}\right)^{\Delta H_i} - 1$ a maximum

U_1 value of U at x_0

u velocity inside boundary layer and parallel to surface

w exponent in formula for viscosity

x distance measured along surface from forward stagnation point

x_0 position on surface at beginning of integration

y distance measured normal to x

α angle between tangent to surface of body of revolution or round channel and axis of revolution

$\beta = r_t\theta + \Omega \cos \alpha$

β_1 value of β at x_0

γ ratio of specific heat at constant pressure to specific heat at constant volume

δ nominal thickness of boundary layer

δ^* displacement thickness $\left[\int_0^\delta \left(1 - \frac{\rho u}{\rho_\delta U}\right) dy\right]$

θ momentum thickness $\left[\int_0^\delta \frac{\rho u}{\rho_\delta U} \left(1 - \frac{u}{U}\right) dy\right]$

θ_1 value of θ at x_0

[redacted]

$\lambda = \sqrt{\dfrac{\phi}{(U/U_o)^2}}$

μ	coefficient of viscosity
μ_o	free-stream viscosity
υ	kinematic viscosity at outer edge of boundary layer
υ_o	kinematic viscosity in free stream
ρ	density
ρ_o	free-stream density
ρ_δ	density at edge of boundary layer
τ_o	surface shearing stress

$\phi = 1 + \dfrac{2}{(\gamma - 1)M_o^2}$

$\psi = r_t\theta - \Omega \cos \alpha$

ψ_1 value of ψ at x_0

$\Omega = \int_0^\delta \frac{\rho u}{\rho_\delta U}\left(1 - \frac{u}{U}\right)y\,dy$

$\Omega^* = \int_0^\delta \left(1 - \frac{\rho u}{\rho_\delta U}\right)y\,dy$

Subscripts:

c	compressible flow
i	incompressible flow

ANALYSIS

Momentum equation for compressible flow about a two-dimensional body.- The boundary-layer momentum equation for two-dimensional compressible flow (reference 10, p.132) is given, when the static-pressure variation across the boundary layer is negligible, as

$$\frac{\partial}{\partial x}\int_0^\delta \rho u^2\, dy - U\frac{\partial}{\partial x}\int_0^\delta \rho u\, dy = -\tau_o - \delta\frac{\partial p}{\partial x} \tag{1}$$

From the equation of motion for compressible, inviscid flow the relation between the velocity and pressure derivatives outside the boundary layer may be written for convenience as

$$\frac{\partial \rho_\delta U^2}{\partial x} - U\frac{\partial \rho_\delta U}{\partial x} = -\frac{\partial p}{\partial x}$$

Then by use of the equation for $\frac{\partial p}{\partial x}$, the definition for the momentum thickness

$$\theta = \int_0^\delta \frac{\rho u}{\rho_\delta U}\left(1 - \frac{u}{U}\right) dy$$

and the definition for the displacement thickness

$$\delta^* = \int_0^\delta \left(1 - \frac{\rho u}{\rho_\delta U}\right) dy$$

equation (1) can be written in the form given in reference 1 as

$$\frac{\partial \theta}{\partial x} + \theta\left(\frac{H_c + 2}{U}\frac{\partial U}{\partial x} + \frac{1}{\rho_\delta}\frac{\partial \rho_\delta}{\partial x}\right) = \frac{\tau_o}{\rho_\delta U^2} \tag{2}$$

where $H_c = \frac{\delta^*}{\theta}$.

The principles used in the derivation of equation (2) were the conservation of mass and Newton's law of motion. Equation (1) is therefore applicable to both subsonic and supersonic flow. This equation is not, however, to be used for flow through a shock wave because in this case the assumptions of the boundary-layer theory may not be applicable.

[redacted]

<u>Momentum equation for compressible flow about a body of revolution</u>.- The boundary-layer momentum equation for compressible flow about a body of revolution (reference 10, p. 133) can be written, when the static-pressure variation across the boundary layer is negligible, as

$$\frac{\partial}{\partial x}\int_0^\delta \rho u^2 r\,dy - U\frac{\partial}{\partial x}\int_0^\delta \rho u r\,dy = -\tau_o r_t - \frac{\partial p}{\partial x}\int_0^\delta r\,dy \qquad (3)$$

Since

$$r = r_t + y\cos\alpha$$

where r_t and $\cos\alpha$ are dependent on x only (fig. 1), equation (3) may be rewritten as

$$\left(\frac{\partial}{\partial x}\int_0^\delta \rho u^2 r_t\,dy - U\frac{\partial}{\partial x}\int_0^\delta \rho u r_t\,dy\right)$$

$$+\left[\frac{\partial}{\partial x}\int_0^\delta \rho u^2(\cos\alpha)y\,dy - U\frac{\partial}{\partial x}\int_0^\delta \rho u(\cos\alpha)y\,dy\right] =$$

$$-\frac{\partial p}{\partial x}\int_0^\delta r_t\,dy - \frac{\partial p}{\partial x}\int_0^\delta (\cos\alpha)y\,dy - r_t\tau_o$$

If θ and Ω are defined by the relations

$$r_t\theta\rho_\delta U^2 = \int_0^\delta \rho u(U - u)r_t\,dy$$

and

$$\Omega(\cos\alpha)\rho_\delta U^2 = \int_0^\delta \rho u(U - u)(\cos\alpha)y\,dy$$

and the equation of motion for compressible, inviscid flow

$$\frac{\partial p}{\partial x} = -\rho_\delta U\frac{\partial U}{\partial x}$$

is used, then the momentum equation becomes

$$-\frac{\partial U}{\partial x}\left(\int_0^{\delta} \rho_\delta U r_t\,dy - \int_0^{\delta} \rho u r_t\,dy\right) - \frac{\partial}{\partial x}\left(r_t\theta\rho_\delta U^2\right)$$

$$-\frac{\partial U}{\partial x}\left[\int_0^{\delta} \rho_\delta U(\cos\alpha)y\,dy - \int_0^{\delta} \rho u(\cos\alpha)y\,dy\right]$$

$$-\frac{\partial}{\partial x}\left[\Omega(\cos\alpha)\rho_\delta U^2\right] = -r_t\tau_o$$

Then δ^* and Ω^* may be defined by the relations

$$r_t\delta^*\rho_\delta U = \int_0^{\delta} \left(\rho_\delta U - \rho u\right)r_t\,dy$$

and

$$\Omega^*(\cos\alpha)\rho_\delta U = \int_0^{\delta} \left(\rho_\delta U - \rho u\right)(\cos\alpha)y\,dy$$

so that the momentum equation becomes

$$\frac{\partial}{\partial x}\left(r_t\theta + \Omega\cos\alpha\right) + \frac{1}{U}\frac{\partial U}{\partial x}\left[r_t\theta\left(H_c + 2\right) + \Omega(\cos\alpha)\left(\frac{\Omega^*}{\Omega} + 2\right)\right]$$

$$+\frac{1}{\rho_\delta}\frac{\partial\rho_\delta}{\partial x}\left(r_t\theta + \Omega\cos\alpha\right) = \frac{r_t\tau_o}{\rho_\delta U^2} \qquad (4)$$

In order to permit integration of the momentum equation for compressible flow about a body of revolution, equation (4) should be written in the same form as that for two-dimensional compressible flow (equation (2)). The approximation is therefore made that for flow on the body of revolution

$$\left[r_t\theta\left(H_c + 2\right) + \Omega(\cos\alpha)\left(\frac{\Omega^*}{\Omega} + 2\right)\right] = \left(H_c + 2\right)\left(r_t\theta + \Omega\cos\alpha\right)K$$

or

$$\frac{1 + \dfrac{\Omega \cos \alpha}{r_t\theta}\left(\dfrac{\dfrac{\Omega^*}{\Omega} + 2}{H_c + 2}\right)}{1 + \dfrac{\Omega \cos \alpha}{r_t\theta}} = K$$

In order to determine the magnitude of the ratio $\dfrac{\dfrac{\Omega^*}{\Omega} + 2}{H_c + 2}$, the assumption is made that the velocity distributions through the boundary layer may be approximated by power curves of the type

$$\frac{u}{U} = \left(\frac{y}{\delta}\right)^{1/m}$$

When the flow is incompressible, the definitions of Ω^*, Ω, and H_i may be used to obtain

$$\frac{\Omega^*}{\Omega} = \frac{1 + m}{m}$$

and

$$H_i = \frac{2 + m}{m}$$

Therefore

$$\frac{\dfrac{\Omega^*_i}{\Omega_i} + 2}{H_i + 2} = \frac{3m + 1}{3m + 2}$$

for incompressible flow.

From

$$\int_0^1 F\left(y^{2/p}\right) y \, dy = \frac{1}{2} \int_0^1 F\left(y^{1/p}\right) \, dy$$

it can be shown that

$$\left(\frac{\Omega^*}{\Omega}\right)_{m=m'} = H_{m=2m'}$$

for compressible as well as for incompressible flow. By use of

$$\left(\frac{\Omega^*}{\Omega}\right)_{m=m'} = H_{m=2m'}$$

and figure 2, which is discussed later in connection with the effect of compressibility on H_c, the ratio $\dfrac{\frac{\Omega^*}{\Omega} + 2}{H_c + 2}$ can be evaluated for compressible flow. For all cases of unseparated flow, the value of the ratio $\dfrac{\frac{\Omega^*}{\Omega} + 2}{H_c + 2}$ differs from unity by less than 20 percent and, for most cases, by roughly 10 percent. Thus, even when $\dfrac{\Omega \cos \alpha}{r_t\theta}$ is not a small fraction, the approximation that $K = 1$ is not far from true. When, in addition, it is noted that

$$\frac{\Omega \cos \alpha}{r_t\theta} < \frac{\delta}{r_t}$$

and is therefore small over most of the body, the approximation that $K = 1$ is permissible. If $K = 1$ is used in equation (4), it becomes

$$\frac{\partial}{\partial x}\left(r_t\theta + \Omega \cos \alpha\right) + \frac{H_c + 2}{U}\frac{\partial U}{\partial x}\left(r_t\theta + \Omega \cos \alpha\right) + \frac{1}{\rho_\delta}\frac{\partial \rho_\delta}{\partial x}\left(r_t\theta + \Omega \cos \alpha\right) = \frac{r_t\tau_o}{\rho_\delta U^2}$$

or by substitution of β for $r_t\theta + \Omega \cos \alpha$

$$\frac{\partial \beta}{\partial x} + \beta\left(\frac{H_c + 2}{U}\frac{\partial U}{\partial x} + \frac{1}{\rho_\delta}\frac{\partial \rho_\delta}{\partial x}\right) = \frac{r_t\tau_o}{\rho_\delta U^2} \qquad (5)$$

This equation for incompressible flow has been given in reference 3.

The principles used in the derivation of equation (5) were the conservation of mass and Newton's law of motion. The approximation was made that for flow on the body $K = 1$.

Momentum equation for compressible flow over inner surface of a round channel.- When the flow in a round channel is such that the flow throughout the boundary layer is approximately parallel to the wall and a region exists outside the boundary layer in which viscosity has no effect, the boundary-layer momentum equation for this flow is the same as equation (3). In the momentum equation for flow over the inner surface of a round channel, the pressure change across the boundary layer is assumed to be negligible. All of the symbols in the momentum equation for compressible flow over the inner surface of a round channel have their meanings unchanged except for the distance y, which is positive when measured inward from the wall so that

$$r = r_t - y \cos \alpha$$

The derivation of the momentum equation for flow over the inner surface of a round channel follows the same procedure and involves the same assumptions as the derivation of equation (5). The derived equation is

$$\frac{\partial \psi}{\partial x} + \psi\left(\frac{H_c + 2}{U}\frac{\partial U}{\partial x} + \frac{1}{\rho_\delta}\frac{\partial \rho_\delta}{\partial x}\right) = \frac{r_t \tau_o}{\rho_\delta U^2} \qquad (6)$$

where

$$\psi = r_t \theta - \Omega \cos \alpha$$

Effect of compressibility on H_c.- Since the ratio of the displacement thickness to the momentum thickness H_c occurs in the momentum equation, the effect of compressibility upon this ratio should be considered. The equation for H_c is

$$H_c = \frac{\delta^*}{\theta} = \frac{\int_0^\delta \left(1 - \frac{\rho u}{\rho_\delta U}\right) dy}{\int_0^\delta \frac{\rho u}{\rho_\delta U}\left(1 - \frac{u}{U}\right) dy}$$

The density variation through the boundary layer can be obtained from the velocity distribution through the boundary layer by restricting the treatment to the case in which there is no heat flow through the surface and the effective Prandtl number is equal to unity. The equation

$$c_pT + \frac{u^2}{2} = \text{constant}$$

is then applicable to flow in the boundary layer. When the additional restriction is made that the static-pressure variation through the boundary layer is negligible, the expression for $\frac{\rho}{\rho_\delta}$ becomes

$$\frac{\rho}{\rho_\delta} = \frac{1 + \left[1 - \left(\frac{U}{U_o}\right)^2\right]\frac{\gamma - 1}{2} M_o^2}{1 + \left[1 - \left(\frac{u}{U}\right)^2\left(\frac{U}{U_o}\right)^2\right]\frac{\gamma - 1}{2} M_o^2}$$

and the expression for θ becomes

$$\frac{\theta}{\delta} = \int_0^1 \frac{1 + \left[1 - \left(\frac{U}{U_o}\right)^2\right]\frac{\gamma - 1}{2} M_o^2}{1 + \left[1 - \left(\frac{u}{U}\right)^2\left(\frac{U}{U_o}\right)^2\right]\frac{\gamma - 1}{2} M_o^2} \frac{u}{U}\left(1 - \frac{u}{U}\right) d\frac{y}{\delta}$$

which can be reduced to

$$\frac{\theta}{\delta} = \left[\frac{1 + \frac{\gamma - 1}{2} M_o^2}{\left(\frac{U}{U_o}\right)^2 \frac{\gamma - 1}{2} M_o^2} - 1\right]\int_0^1 \frac{\frac{u}{U}\left(1 - \frac{u}{U}\right) d\frac{y}{\delta}}{\frac{1 + \frac{\gamma - 1}{2} M_o^2}{\left(\frac{U}{U_o}\right)^2 \frac{\gamma - 1}{2} M_o^2} - \left(\frac{u}{U}\right)^2}$$

[redacted]

Let

$$\frac{1 + \frac{\gamma - 1}{2} M_o^2}{\left(\frac{U}{U_o}\right)^2 \frac{\gamma - 1}{2} M_o^2} = \lambda^2$$

or

$$\lambda^2 = \frac{\phi}{\left(\frac{U}{U_o}\right)^2}$$

where

$$\phi = 1 + \frac{2}{(\gamma - 1) M_o^2}$$

then

$$\frac{\theta}{\delta} = \left(\lambda^2 - 1\right) \int_0^1 \frac{\frac{u}{U}\left(1 - \frac{u}{U}\right)}{\lambda^2 - \left(\frac{u}{U}\right)^2} d\frac{y}{\delta} \tag{7}$$

By application of the same procedure to the definition of δ^*/δ, then

$$\frac{\delta^*}{\delta} = 1 - \left(\lambda^2 - 1\right) \int_0^1 \frac{\frac{u}{U}}{\lambda^2 - \left(\frac{u}{U}\right)^2} d\frac{y}{\delta} \tag{8}$$

Large values of λ^2 mean small values of M_o.

The assumption was made that

$$\frac{u}{U} = \left(\frac{y}{\delta}\right)^{1/m}$$

and values were chosen for λ^2; δ^*/δ and θ/δ were then calculated for a range of λ between 1.5 and 14.0 with $m = 3$, 4, 5, and 7. The curves of $1/H_c$ are given in figure 2. Since

$$H_i = \frac{2 + m}{m}$$

and since H appears in the momentum equation, the curves of figure 2 are designated by values of H_i rather than m. Power curves are used merely for convenience in computing the effect of compressibility on H.

The value of H_c for the Blasius flat-plate profile for laminar boundary layers (reference 2, p. 88) has been computed for various values of λ and a Prandtl number of unity by use of equations (7) and (8). The computations were repeated with the velocity distribution for laminar flow over a flat plate at $M_o = 2$ (reference 11). When H_c was plotted against λ, the results of both computations were practically identical for small values of λ. Only the results obtained for the Blasius profile are therefore presented in figure 3.

Equations (7) and (8) show that although the velocity distribution through the boundary layer is assumed to be independent of position along the surface, H_c may vary with surface position because of its dependence on λ. For integration of the momentum equations, the dependence of H_c on λ may be taken into account by approximating the curves of figures 2 and 3 by the equation

$$H_c = \frac{a}{\lambda - 1} + H_i$$

$$= \frac{a}{\left(\dfrac{\sqrt{\phi}}{\dfrac{U}{U_o}}\right) - 1} + H_i \tag{9}$$

The values of a chosen to fit the curves of figures 2 and 3 with sufficient accuracy over most of the range of λ are plotted against H_i in figure 4.

<u>Integration of momentum equation for two-dimensional flow.</u>- Before equation (2) can be integrated, $(H_c + 2)$, $\dfrac{1}{\rho_\delta}\dfrac{\partial \rho_\delta}{\partial x}$, and $\dfrac{\tau_o}{\rho_\delta U^2}$ should be replaced by functions of the velocity distribution over the body, the free-stream Mach number, and the momentum thickness. The term $H_c + 2$ is replaced by its equivalent $\dfrac{a}{\left(\dfrac{\sqrt{\phi}}{\dfrac{U}{U_o}}\right) - 1} + (H_i + 2)$. By use

of the gas law

$$\rho_\delta = \frac{p}{RT_\delta}$$

the equation of motion

$$\frac{\partial p}{\partial x} = - \rho_\delta U \frac{\partial U}{\partial x}$$

the Bernoulli equation for compressible flow

$$c_p T_\delta + \frac{U^2}{2} = c_p T_o + \frac{U_o^2}{2}$$

and the equation for the velocity of sound

$$c_o^2 = (\gamma - 1)\, c_p T_o$$

the term $\frac{1}{\rho_\delta}\frac{\partial \rho_\delta}{\partial x}$ can be written as

$$\frac{1}{\rho_\delta}\frac{\partial \rho_\delta}{\partial x} = \frac{-2\left(\frac{U}{U_o}\right)\frac{\partial\left(\frac{U}{U_o}\right)}{\partial x}}{(\gamma - 1)\left[\phi - \left(\frac{U}{U_o}\right)^2\right]} \tag{10}$$

The local skin-friction coefficient $\frac{\tau_o}{\rho_\delta U^2}$ is expressed as a power function of the local Reynolds number based on the boundary-layer momentum thickness by

$$\frac{\tau_o}{\rho_\delta U^2} = \frac{k}{R_\theta{}^n} = k\left(\frac{\upsilon}{U}\right)^n \theta^{-n} \tag{11}$$

By analogy with the work of reference 11, the viscosity and density used to calculate R_θ are those at the outer edge of the boundary layer. The viscosity at the outer edge of the boundary layer is assumed to be given by

$$\frac{\mu}{\mu_o} = \left(\frac{T_\delta}{T_o}\right)^w \tag{12}$$

where $w = 0.768$ for air (reference 11).

The density at the outer edge of the boundary layer is given by

$$\frac{\rho_o}{\rho_\delta}=\left[\frac{\phi-\left(\frac{U}{U_o}\right)^2}{\phi-1}\right]^{\frac{1}{1-\gamma}} \tag{13}$$

in which the flow outside the boundary layer is assumed to be adiabatic. Equations (12) and (13) are used to give the kinematic viscosity ν in equation (11) as a function of the velocity distribution over the body and the free-stream Mach number. Equation (2) may then be written as

$$\frac{\partial\theta}{\partial x}+\theta\left\{\frac{a}{\sqrt{\phi}-\frac{U}{U_o}}\,\frac{\partial\frac{U}{U_o}}{\partial x}+\frac{H_1+2}{\left(\frac{U}{U_o}\right)}\,\frac{\partial\left(\frac{U}{U_o}\right)}{\partial x}-\frac{2\left(\frac{U}{U_o}\right)\frac{\partial\left(\frac{U}{U_o}\right)}{\partial x}}{(\gamma-1)\left[\phi-\left(\frac{U}{U_o}\right)^2\right]}\right\}=\frac{k\nu^n}{\left(\frac{U}{U_o}\right)^n\theta^n U_o{}^n}$$

This equation is a differential equation of the Bernoulli type. When it is made linear and integrated by standard methods, the result is

$$\left(\frac{\theta}{L}\right)_{x/L}=\frac{\left[\sqrt{\phi}-\left(\frac{U}{U_o}\right)_{x/L}\right]^a}{\left[\left(\frac{U}{U_o}\right)_{x/L}\right]^{(H_1+2)}\left[\phi-\left(\frac{U}{U_o}\right)_{x/L}^{2}\right]^{\frac{1}{\gamma-1}}}\left\{\frac{\left[\frac{k(1+n)}{R_L{}^n}\right]}{(\phi-1)^{n\left(w-\frac{1}{\gamma-1}\right)}}\int_{x_0/L}^{x/L}\frac{\left(\frac{U}{U_o}\right)^{\left[(H_1+1)(1+n)+1\right]}\left[\phi-\left(\frac{U}{U_o}\right)^2\right]^{\left(nw+\frac{1}{\gamma-1}\right)}}{\left(\sqrt{\phi}-\frac{U}{U_o}\right)^{a(1+n)}}\,d\frac{x}{L}\right.$$

$$\left.+\frac{\left(\frac{\theta_1}{L}\right)^{1+n}\left(\frac{U_1}{U_o}\right)^{(H_1+2)(1+n)}\left[\phi-\left(\frac{U_1}{U_o}\right)^2\right]^{\frac{1+n}{\gamma-1}}}{\left[\sqrt{\phi}-\frac{U_1}{U_o}\right]^{a(1+n)}}\right\}^{\frac{1}{1+n}} \tag{14}$$

[redacted]

where θ_1/L and U_1/U_o are the values of θ/L and U/U_o at $\frac{x}{L} = \frac{x_o}{L}$. The value of ϕ is always greater than 1. By use of the Bernoulli equation for compressible flow, $\left(\frac{U}{U_o}\right)^2$ can be shown to be always less than ϕ. When M_o approaches zero, equation (14) becomes equation (1) of reference 6, that is, the integrated momentum equation for incompressible flow.

Integration of momentum equation for flow over a body of revolution.- The assumptions concerning H_c are the same as those for the two-dimensional case. The equations for $\frac{\mu}{\mu_o}$, $\frac{\rho_o}{\rho_\delta}$, and $\frac{1}{\rho_\delta}\frac{\partial \rho_\delta}{\partial x}$ are also the same as for the two-dimensional case. The expression for the skin friction, however, involves the approximation that θ can be replaced by $\frac{\beta}{r_t}$. This approximation is equivalent to neglecting the term $\frac{\Omega}{r_t}\cos\alpha$ in the equation

$$\theta = \frac{\beta}{r_t} - \frac{\Omega}{r_t}\cos\alpha$$

Since the value of $\frac{\Omega}{r_t}\cos\alpha$ is always less than $\frac{\delta}{r_t}\theta$, it follows that the approximation is justified only in cases in which $\frac{\delta}{r_t}$ is a small fraction of unity. In regions near the tail, therefore, the accuracy of the approximation for the skin friction may be expected to decrease. The momentum equation for flow over a body of revolution (equation (5)) indicates, however, that the contribution of the skin friction to the boundary-layer thickness becomes less important as the tail is approached. The approximation that θ can be replaced by $\frac{\beta}{r_t}$ is therefore allowable and the term $\frac{r_t\tau_o}{\rho_\delta U^2}$ in equation (5) may be written

$$\frac{r_t\tau_o}{\rho_\delta U^2} = \frac{k r_t^{\,n+1} \nu^n}{\beta^n U^n} \tag{15}$$

When the terms H_c, $\frac{1}{\rho_\delta}\frac{\partial\rho_\delta}{\partial x}$, and $\frac{r_t\tau_o}{\rho_\delta U^2}$ are replaced by equations (9), (10), and (15), then equation (5) becomes

$$\frac{\partial\beta}{\partial x} + \beta\left\{\frac{a}{\sqrt{\phi} - \frac{U}{U_o}}\frac{\partial\frac{U}{U_o}}{\partial x} + \frac{H_1 + 2}{\left(\frac{U}{U_o}\right)}\frac{\partial\frac{U}{U_o}}{\partial x} - \frac{2\left(\frac{U}{U_o}\right)\partial\frac{U}{U_o}}{(\gamma - 1)\left[\phi - \left(\frac{U}{U_o}\right)^2\right]\partial x}\right\} = \frac{r_t^{1+n}k\upsilon^n}{\left(\frac{U}{U_o}\right)^n U_o^{\,n}\beta^n}$$

This equation is a differential equation of the Bernoulli type. When it is made linear and integrated by standard methods, the result is

$$\left(\frac{\beta}{r_{t_{max}}^{2}}\right)_{x/L} = \frac{\left[\sqrt{\phi} - \left(\frac{U}{U_o}\right)_{x/L}\right]^a}{\left[\left(\frac{U}{U_o}\right)_{x/L}\right]^{(H_1+2)}\left[\phi - \left(\frac{U}{U_o}\right)_{x/L}^2\right]^{\frac{1}{\gamma-1}}}\left\{\frac{\frac{k(1+n)}{R_L^{\,n}}\left(\frac{L}{r_{t_{max}}}\right)^{1+n}}{(\phi - 1)^n\left(w - \frac{1}{\gamma-1}\right)}\int_{x_0/L}^{x/L}\frac{\left(\frac{U}{U_o}\right)^{\left[(H_1+1)(1+n)+1\right]}\left[\phi - \left(\frac{U}{U_o}\right)^2\right]^{\left(nw+\frac{1}{\gamma-1}\right)}\left(\frac{r_t}{r_{t_{max}}}\right)^{1+n}}{\left[\sqrt{\phi} - \left(\frac{U}{U_o}\right)\right]^{a(1+n)}}\,d\frac{x}{L}\right.$$

$$\left. + \frac{\left(\frac{\beta_1}{r_{t_{max}}^{2}}\right)^{(1+n)}\left(\frac{U_1}{U_o}\right)^{(H_1+2)(1+n)}\left[\phi - \left(\frac{U_1}{U_o}\right)^2\right]^{\frac{1+n}{\gamma-1}}}{\left[\sqrt{\phi} - \left(\frac{U_1}{U_o}\right)\right]^{a(1+n)}}\right\}^{\frac{1}{1+n}} \qquad (16)$$

where $\frac{\beta_1}{r_{t_{max}}^{2}}$ and U_1/U_o are the values of $\frac{\beta}{r_{t_{max}}^{2}}$ and U/U_o at $\frac{x}{L} = \frac{x_0}{L}$.

[redacted]

Integration of momentum equation for flow over inner surface of a round channel.- The equation resulting from the integration of equation (6), when the same procedure and assumptions made to integrate equation (5) are used, may be obtained from equation (16) by replacing β by ψ. In the equation for ψ, the quantities denoting reference conditions, U_o, rt_{max}, and M_o, are the conditions at a convenient reference section of the channel and the length L is any convenient length.

The definite integrals occurring in equations (14) and (16) and the equation for ψ may be evaluated by either an analytical or a graphical method, whichever is more convenient.

DISCUSSION

Before equation (14), equation (16), or the integrated equation containing ψ can be used, it is necessary to know the velocity distribution over the surface, the constants in the skin-friction formula, and the value to choose for H_i. The velocity distribution must be that for the Mach number and Reynolds number for which the computation is being made.

Skin-friction formulas.- For the turbulent boundary layer, a power function of R_θ is used for the skin-friction formula. Because references 7 and 8 indicate no noticeable effect of Mach number on skin friction, a skin-friction formula for low speeds may be chosen and approximated by

$$\frac{\tau_o}{\rho_\delta U^2} = \frac{k}{{R_\theta}^n}$$

as outlined in reference 6. One formula of the required form is that of Falkner (reference 12)

$$\frac{\tau_o}{\rho_\delta U^2} = \frac{0.006535}{{R_\theta}^{1/6}}$$

If the skin-friction data available are given in the form of $c_d \sim R_x$, they may be converted to $\frac{\tau_o}{\rho_\delta U^2} \sim R_\theta$ by use of the relations

$$R_\theta = \frac{c_d}{2}R_x$$

and

$$\frac{\tau_o}{\rho_\delta U^2} = \frac{1}{2}\left(c_d + R_x \frac{dc_d}{dR_x}\right)$$

Laminar boundary layer.- Although equation (14), equation (16), and the equation for ψ have been derived for turbulent boundary layers, these equations may be used for the approximate computation of θ, β, and ψ in laminar boundary layers by the appropriate choice of a value for H_i and of a skin-friction formula. Although by use of the skin-friction relation for the Blasius flat-plate profile the effect of pressure gradient on the skin-friction coefficient is neglected, the error introduced is small so long as the average pressure gradient over the extent of the laminar boundary layer is small. The skin-friction relation then is

$$\frac{\tau_o}{\rho_\delta U^2} = \frac{0.220}{R_\theta}$$

where $k = 0.220$ and $n = 1$. Although a small decrease occurs in k as the Mach number increases (reference 11), the value of k for incompressible flow (0.220) may be used in view of the approximation already made concerning the skin friction.

Choice of H_i.- In references 1, 3, and 4, the value of the boundary-layer shape parameter H was restricted to 1.4. If equation (14) is used, H_i is given an arbitrary increment ΔH_i, and $\frac{\theta_1}{L} = 0$, then the change in $\left(\frac{\theta}{L}\right)_x$ may be given by

$$\frac{\Delta\left(\frac{\theta}{L}\right)_x}{\left(\frac{\theta}{L}\right)_x} = \left(\frac{U_\lambda}{U_x}\right)^{\Delta H_i} - 1$$

where U_λ lies between the maximum and minimum velocity in the interval x, x_0. The term $\left(\frac{U_\lambda}{U_x}\right)^{\Delta H_i}$ has been

[redacted]

evaluated for incompressible flow and a straight-line velocity distribution

$$U = U_{x_0} \pm bx$$

The expression for $\left(\frac{U_\lambda}{U_x}\right)^{\Delta H_i}$ becomes

$$\left(\frac{U_\lambda}{U_x}\right)^{\Delta H_i} = \left\{\left[\frac{1 - \left(\frac{U_{x_0}}{U_x}\right)^{\left[(H_i+1)(n+1)+2+\Delta H_i(n+1)\right]}}{1 - \left(\frac{U_{x_0}}{U_x}\right)^{\left[(H_i+1)(n+1)+2\right]}}\right] \left[\frac{(H_i + 1)(n + 1) + 2}{(H_i + 1)(n + 1) + 2 + \Delta H_i(n + 1)}\right]\right\}^{\frac{1}{n+1}} \tag{17}$$

Although equation (17) is for incompressible flow and the expression for $\frac{\Delta(\theta/L)_x}{(\theta/L)_x}$ is only approximately correct for cases in which $\frac{\theta_1}{L} \neq 0$, equation (17) indicates that excessive errors in θ/L are possible if a in equation (9) is made to equal zero and H_i is replaced by H_c and restricted to a constant low-speed value when the actual variation of H_c is large. Increasing the body thickness, the lift coefficient, or the Mach number increases the variation of H_c over the surface.

In the integrated equations the value of H_c may vary over the surface but that of H_i is assumed to be constant. Because H_i can usually be estimated to within 0.2 for cases in which flow separation does not occur, the error in θ/L that may be caused by a poor choice of H_i is unlikely to be more than 10 percent.

When the velocity increases in the direction of flow, the value of H_i may be taken as 1.2. When the velocity decreases in the direction of flow, the value of H_i should be increased from 1.4, for cases in which the total change of velocity is about 20 percent of the initial velocity, to values near 1.7, for cases in which the total change in velocity exceeds 30 percent of the initial velocity. Because the velocity profile for the laminar boundary layer has been restricted to the Blasius flat-plate profile, H_1 is equal to 2.6. For the laminar boundary layer, a then has the value 1.2.

Full thickness of boundary layer.- The full thickness of the turbulent boundary layer in the case of two-dimensional flow may be obtained by use of the relation

$$\frac{\delta}{L} = \left(\frac{\theta}{L}\right)\left(\frac{\delta}{\theta}\right)$$

For the body of revolution, the full thickness of the turbulent boundary layer may be obtained, after $\frac{\beta}{{r_t}_{max}^{\,2}}$ is computed, by use of the relation

$$\frac{\delta}{{r_t}_{max}} = \frac{-\left(\frac{r_t}{{r_t}_{max}}\right)\left(\frac{\theta}{\delta}\right) + \sqrt{\left(\frac{r_t}{{r_t}_{max}}\right)^2\left(\frac{\theta}{\delta}\right)^2 + 4\left(\frac{\beta}{{r_t}_{max}^{\,2}}\right)\left(\frac{\Omega}{\delta^2}\right)\cos\alpha}}{2\left(\frac{\Omega}{\delta^2}\right)\cos\alpha} \tag{18}$$

Curves of θ/δ and Ω/δ^2 against λ are given in figures 5 and 6 for four values of H_1.

The variation of θ/δ for the Blasius flat-plate profile is given in figure 7 for use in computing the full thickness of the laminar boundary layer. For flow about a body of revolution, Ω/δ^2 may be neglected since the laminar boundary layers are usually thin. The expression for the full thickness of the laminar boundary layer on a body of revolution therefore becomes

$$\frac{\delta}{{r_t}_{max}} = \left(\frac{\beta}{{r_t}_{max}^{\,2}}\right)\left(\frac{{r_t}_{max}}{r_t}\right)\left(\frac{\delta}{\theta}\right) \tag{19}$$

For flow over the inner surface of a round channel when the boundary layer is thin, equations (18) and (19) are replaced by

$$\frac{\delta}{{r_t}_{max}} = \frac{\left(\frac{r_t}{{r_t}_{max}}\right)\left(\frac{\theta}{\delta}\right) - \sqrt{\left(\frac{r_t}{{r_t}_{max}}\right)^2\left(\frac{\theta}{\delta}\right)^2 - 4\left(\frac{\psi}{{r_t}_{max}^{\,2}}\right)\left(\frac{\Omega}{\delta^2}\right)\cos\alpha}}{2\left(\frac{\Omega}{\delta^2}\right)\cos\alpha} \tag{20}$$

and

$$\frac{\delta}{r_{t_{max}}} = \left(\frac{\psi}{r_{t_{max}}^{2}}\right)\left(\frac{r_{t_{max}}}{r_t}\right)\left(\frac{\delta}{\theta}\right) \tag{21}$$

In order to obtain a general statement regarding the effect of density changes on the momentum thickness, computations of θ were made for a linear velocity distribution and a range of Mach number. The velocity distribution for $0 \leqq \frac{x}{L} \leqq 1$ was defined by

$$\frac{U}{U_o} = 1 + b\frac{x}{L}$$

The only variable was M_o. The curves of θ against M_o are given in figure 8 for $b = 0$ and ± 0.2. The results indicate that the effect of density variation becomes important only at Mach numbers exceeding unity.

CONCLUSIONS

A comparatively rapid method is presented for the computation of boundary-layer momentum thicknesses for flow over two-dimensional bodies, over bodies of revolution at zero angle of attack, and over the inner surface of round channels all in compressible flow.

The computations indicate that the effect of density changes on the momentum thickness in flows with pressure gradients is small for subsonic flows.

Langley Memorial Aeronautical Laboratory
National Advisory Committee for Aeronautics
Langley Field, Va.

REFERENCES

1. Young, A. D., and Winterbottom, N. E.: Note on the Effect of Compressibility on the Profile Drag of Aerofoils in the Absence of Shock Waves. Rep. No. B. A. 1595, R.A.E. (British), May 1940.

2. Prandtl, L.: The Mechanics of Viscous Fluids. Vol. III of Aerodynamic Theory, div. G, secs. 14 and 17, W. F. Durand, ed., Julius Springer (Berlin), 1935, pp. 84-90 and 102-112.

3. Young, A. D.: The Calculation of the Total and Skin Friction Drags of Bodies of Revolution at Zero Incidence. R. & M. No. 1874, British A.R.C., 1939.

4. Squire, H. B., and Young, A. D.: The Calculation of the Profile Drag of Aerofoils. R. & M. No. 1838, British A.R.C., 1938.

5. von Doenhoff, Albert E., and Tetervin, Neal: Determination of General Relations for the Behavior of Turbulent Boundary Layers. NACA ACR No. 3G13, 1943.

6. Tetervin, Neal: A Method for the Rapid Estimation of Turbulent Boundary-Layer Thicknesses for Calculating Profile Drag. NACA ACR No. L4G14, 1944.

7. Frössel, W.: Flow in Smooth Straight Pipes at Velocities above and below Sound Velocity. NACA TM No. 844, 1938.

8. Theodorsen, Theodore, and Regier, Arthur: Experiments on Drag of Revolving Disks, Cylinders, and Streamline Rods at High Speeds. NACA ACR No. L4F16, 1944.

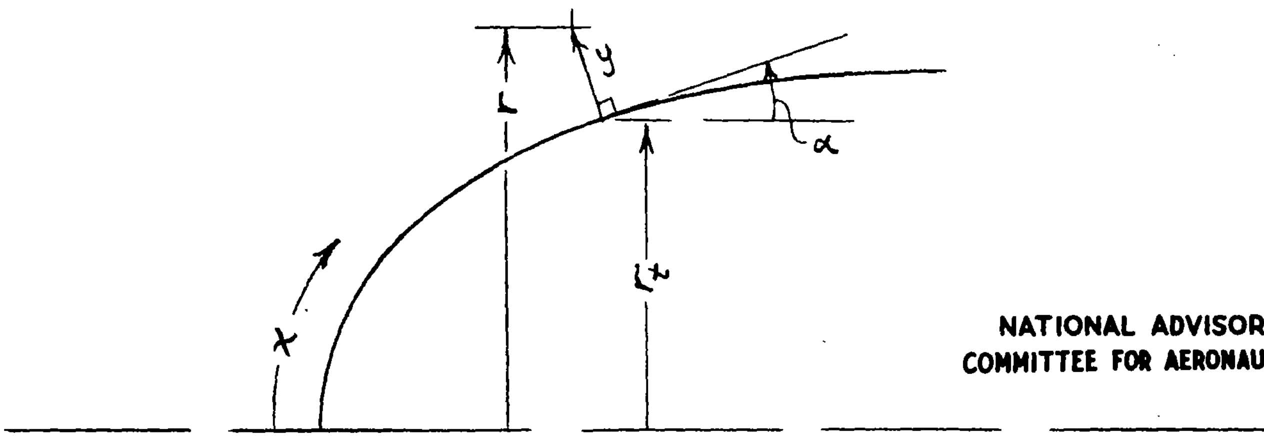
9. Keenan, Joseph H., and Neumann, Ernest P.: Friction in Pipes at Supersonic and Subsonic Velocities. NACA TN No. 963, 1945.

10. Fluid Motion Panel of the Aeronautical Research Committee and Others: Modern Developments in Fluid Dynamics. Vol. I., S. Goldstein, ed., The Clarendon Press (Oxford), 1938.

11. Brainerd, J. G., and Emmons, H. W.: Effect of Variable Viscosity on Boundary Layers, with a Discussion of Drag Measurements. Jour. Appl. Mech., vol. 9, no. 1, March 1942, pp. A-1 - A-6.

12. Falkner, V. M.: A New Law for Calculating Drag. The Resistance of a Smooth Flat Plate with Turbulent Boundary Layer. Aircraft Engineering, vol. XV, no. 169, March 1943, pp. 65-69.

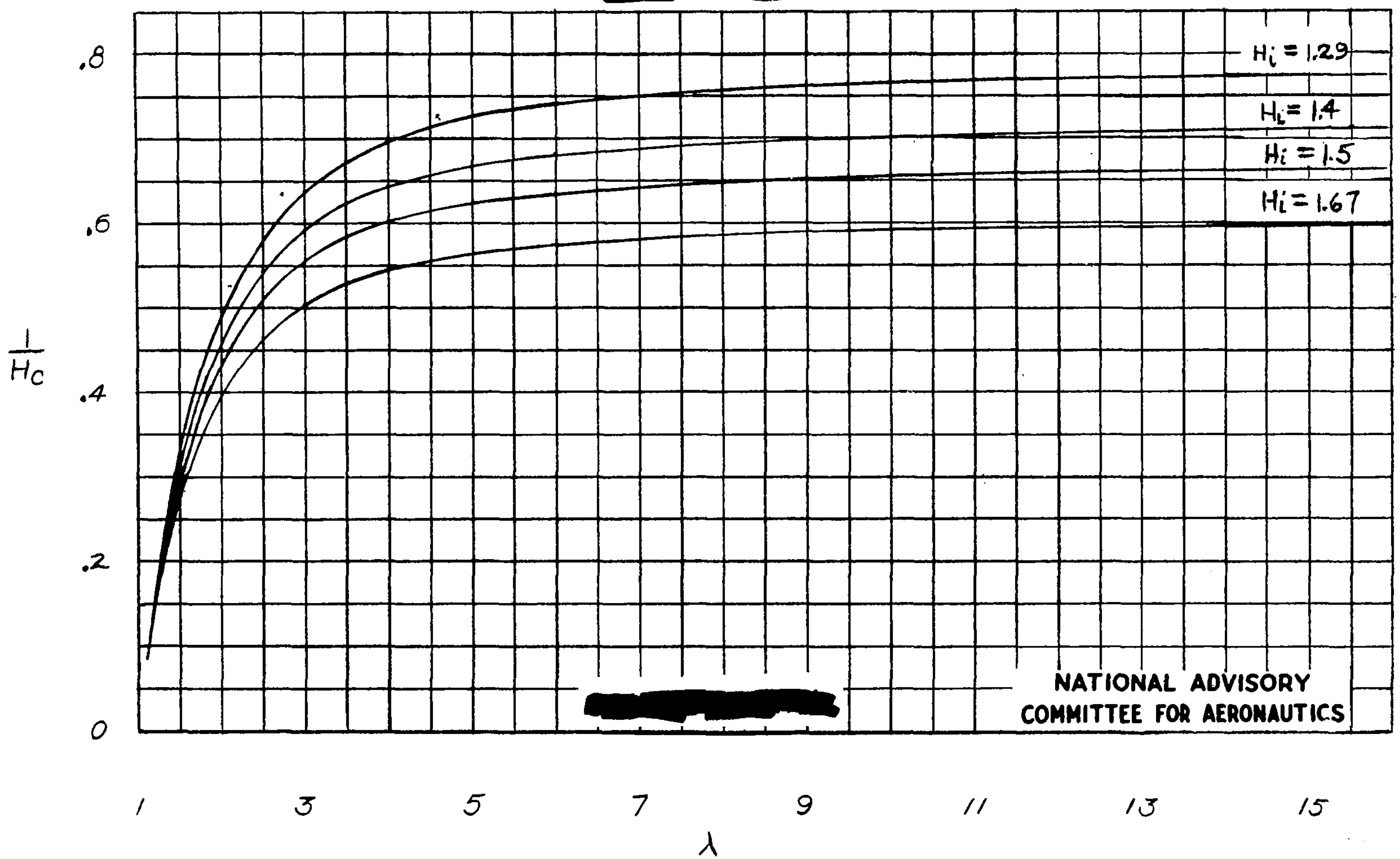
[redacted]



[redacted]

Figure 1.- Coordinates on body of revolution.

[redacted]



[redacted]

Figure 2.- Variation of $1/H_c$ with λ for various values of H_i.

[redacted]

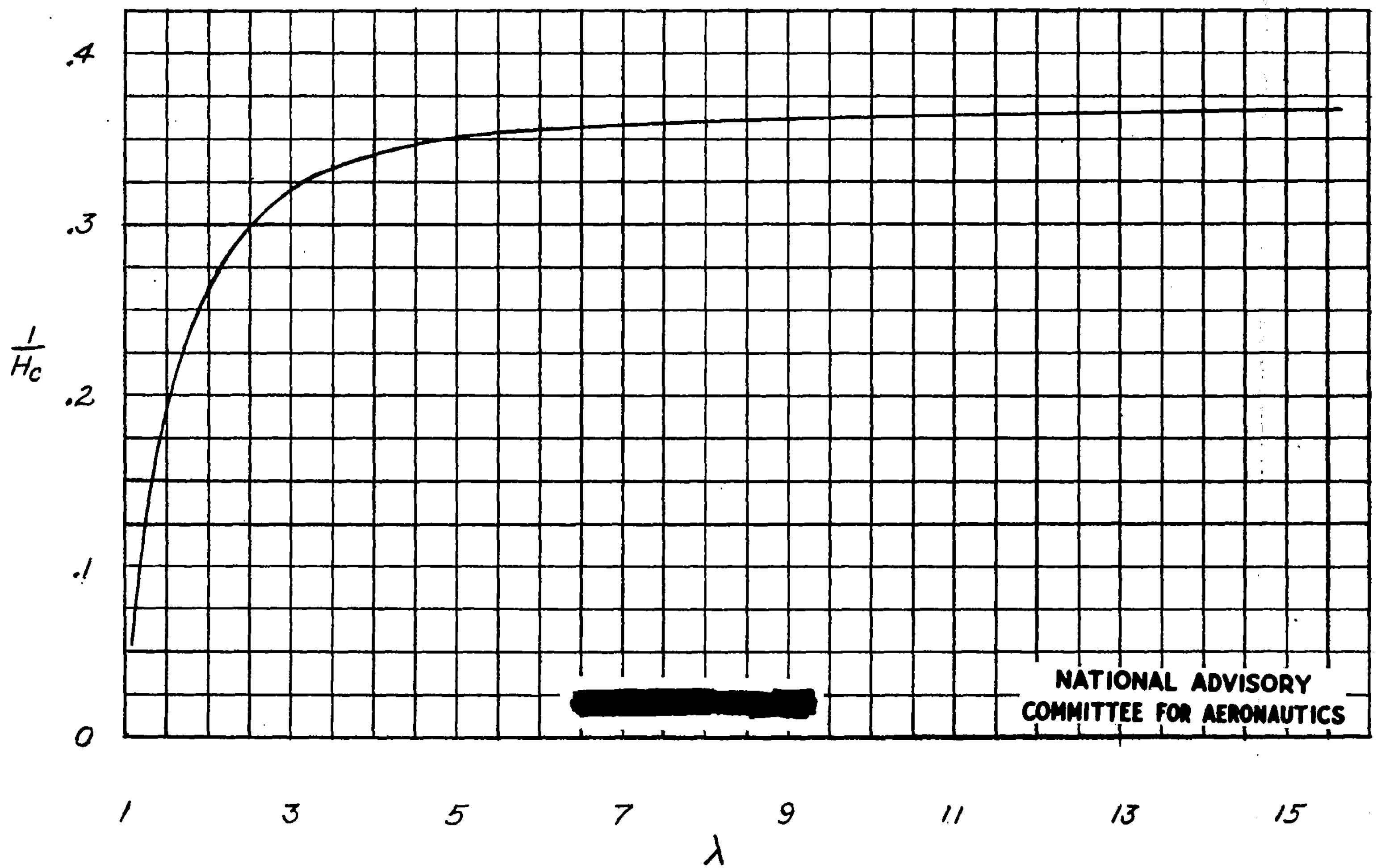


Figure 3.- Variation of $1/H_c$ with λ for the Blasius flat-plate velocity distribution.

[redacted]

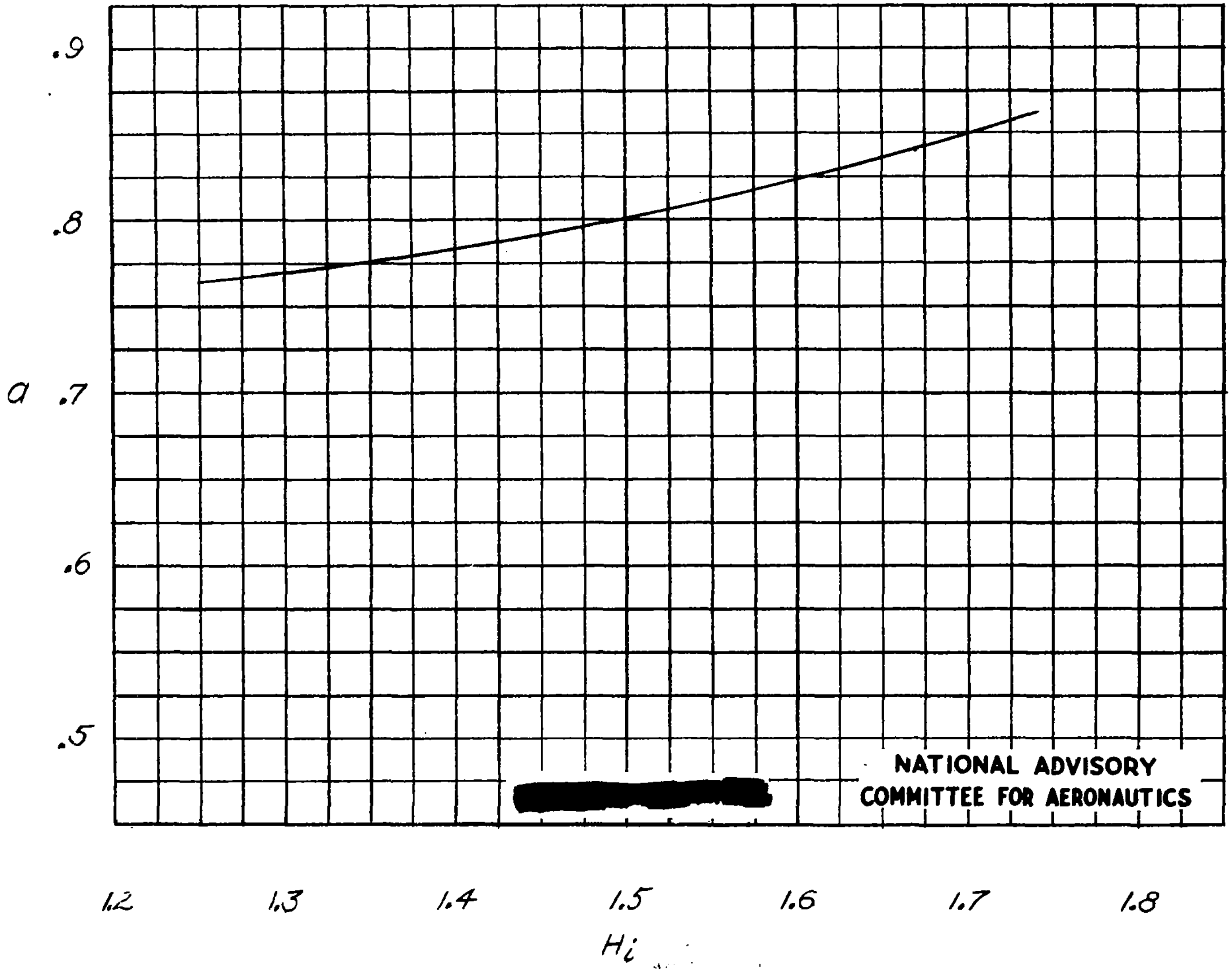


Figure 4.- Variation of a with H_1.

[redacted]

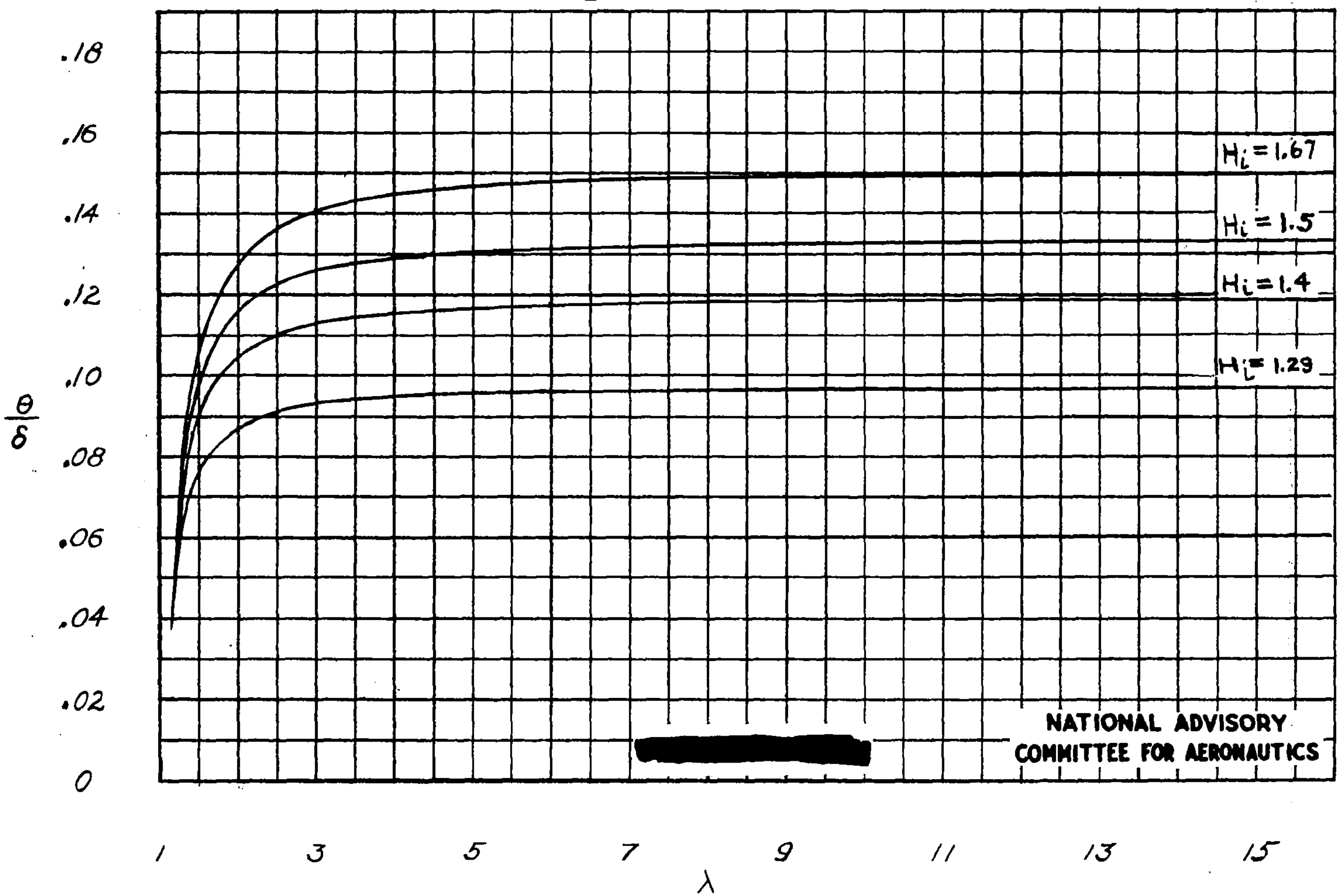


Figure 5.- Variation of θ/δ with λ for various values of H_i.

[redacted]

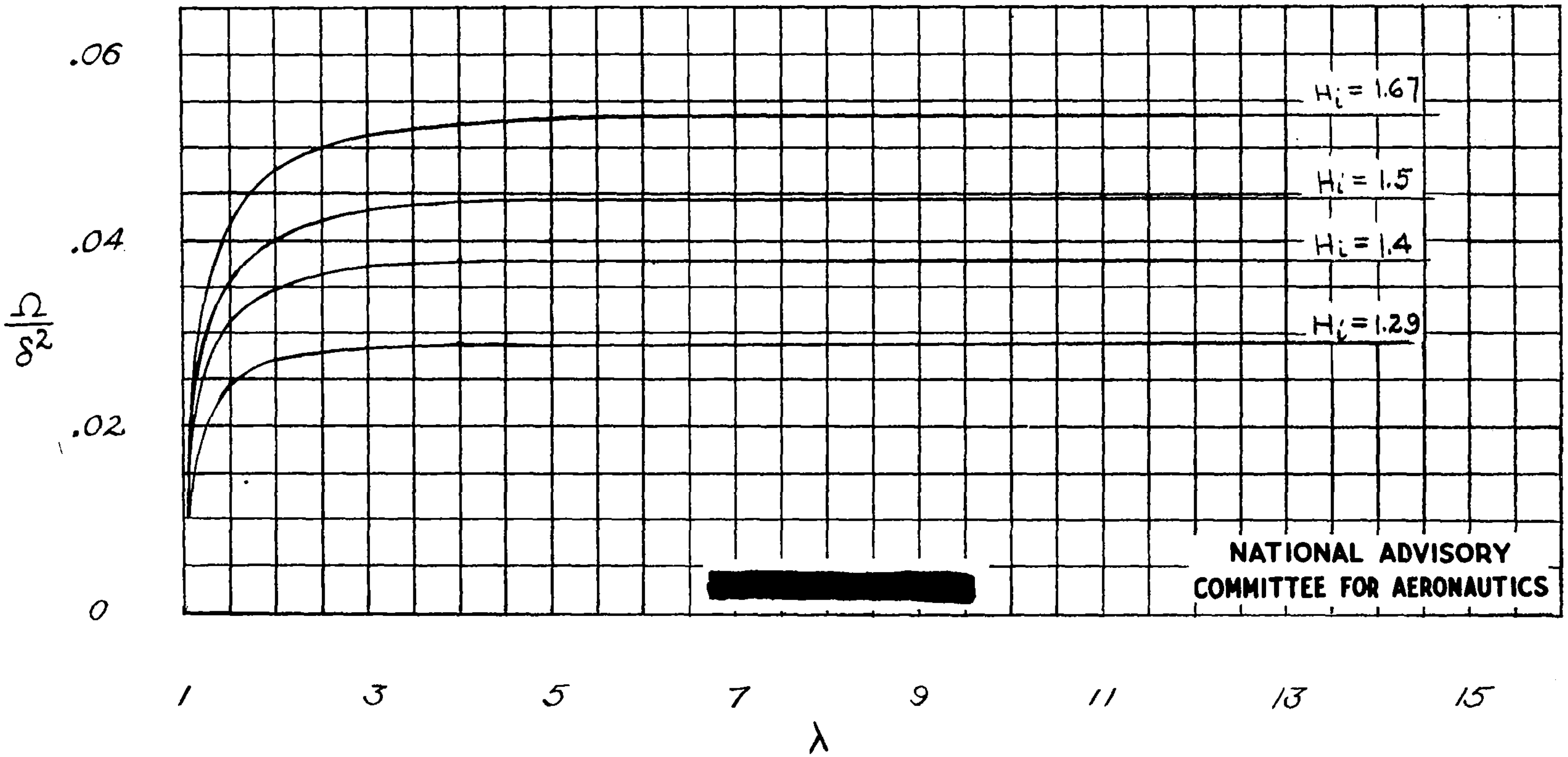


Figure 6.- Variation of Ω/δ^2 with λ for various values of H_i.

[redacted]

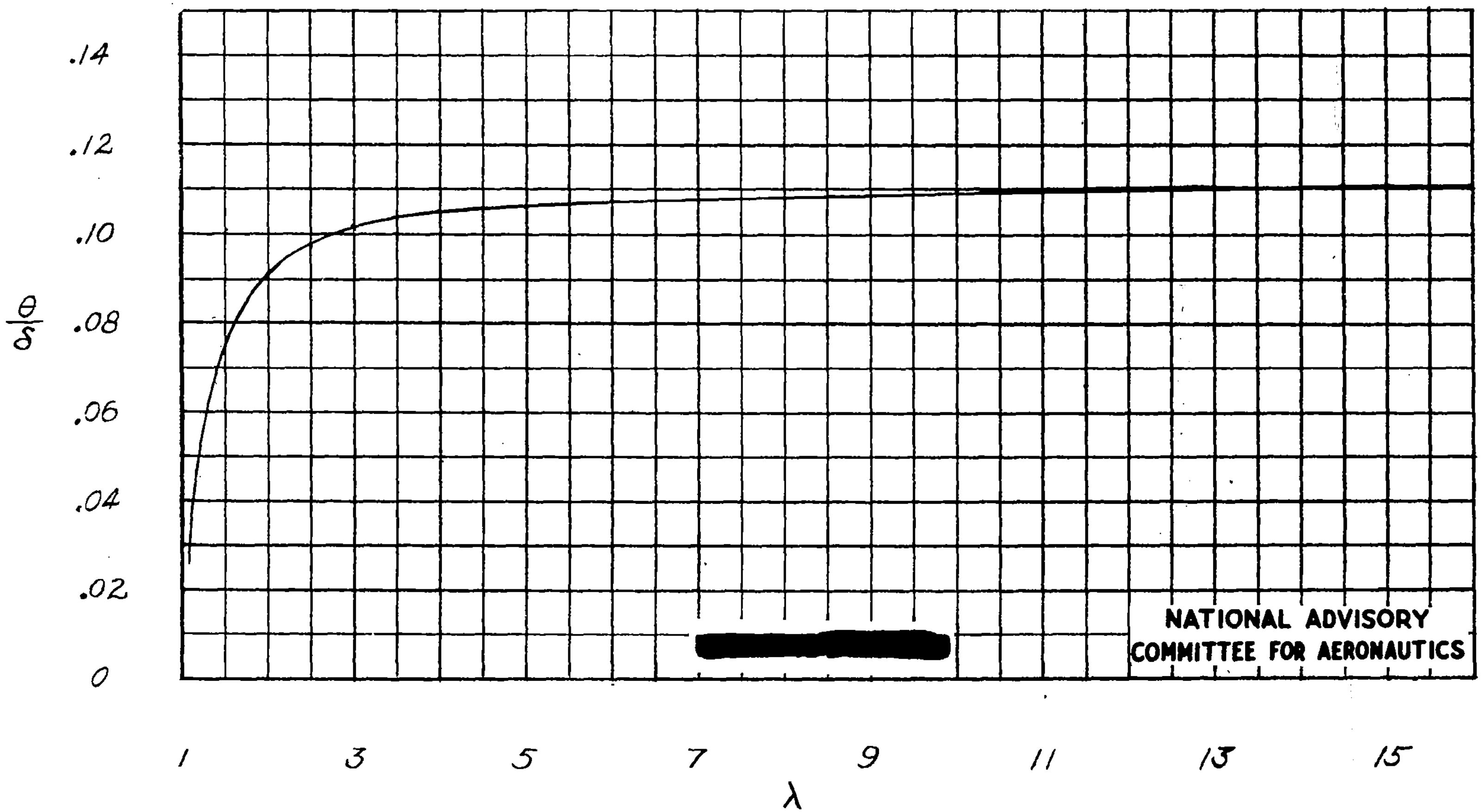
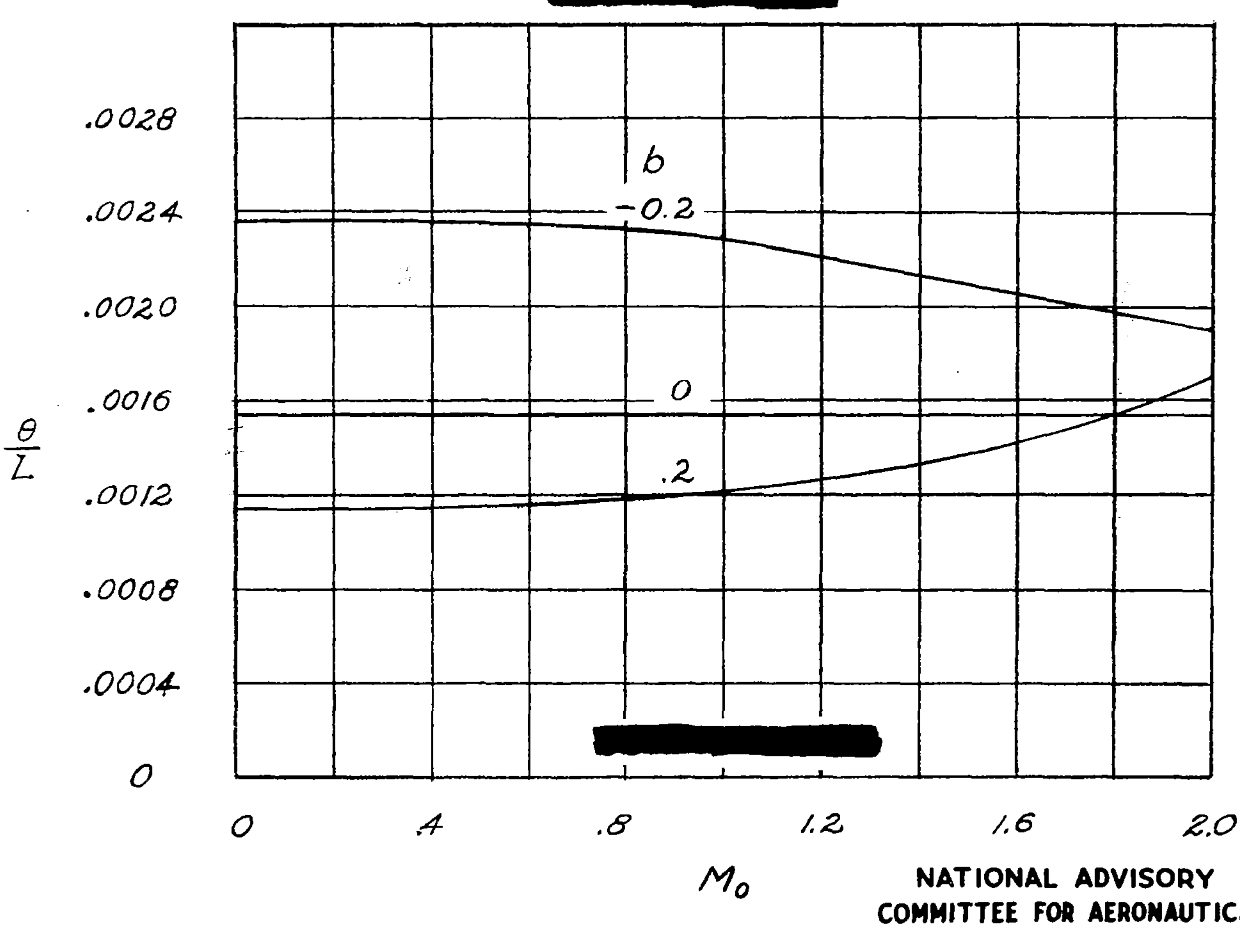


Figure 7.- Variation of θ/δ for the Blasius flat-plate profile with λ.

[redacted]



NATIONAL ADVISORY
COMMITTEE FOR AERONAUTICS

Figure 8.- Variation of θ/L with M_o for velocity distribution given by $\frac{U}{U_o} = 1 + b\frac{x}{L}$; $0 < x/L < 1$. Values of constants in equation (14): $k = 0.006535$; $n = 1/6$; $R_L = 10^7$; $w = 0.768$; $\gamma = 1.4$; $H_i = 1.4$; $a = 0.78$; $\frac{\theta_i}{L} = 0$.